Math 600 Day 6: Abstract Smooth Manifolds

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Outline

1. Transition to abstract smooth manifolds
   - Partitions of unity on differentiable manifolds.

A Word About Last Time

Theorem

(Sard's Theorem) The set of critical values of a smooth map always has measure zero in the receiving space.

Theorem

Let $A \subset \mathbb{R}^n$ be open and let $f : A \to \mathbb{R}^p$ be a smooth function whose derivative $f'(x)$ has maximal rank p whenever $f(x) = 0$. Then $f^{-1}(0)$ is a $(n - p)$-dimensional manifold in $\mathbb{R}^n$.

Transition to abstract smooth manifolds

Up to this point, we have viewed smooth manifolds as subsets of Euclidean spaces, and in that setting have defined:

1. coordinate systems
2. manifolds-with-boundary
3. tangent spaces
4. differentiable maps

and stated the **Chain Rule** and **Inverse Function Theorem**.

Now we want to define **differentiable (= smooth) manifolds** in an abstract setting, without assuming they are subsets of some Euclidean space.

Many differentiable manifolds come to our attention this way. Here are just a few examples:

1. tangent and cotangent bundles over smooth manifolds
2. Grassmann manifolds
3. manifolds built by "surgery" from other manifolds

Our starting point is the definition of a **topological manifold of dimension n** as a topological space M^n in which each point has an open neighborhood homeomorphic to $\mathbb{R}^n$ (**locally Euclidean** property), and which, in addition, is **Hausdorff** and **second countable**.

We recall that **Hausdorff** means that any two distinct points of the space have disjoint open neighborhoods, and **second countable** means that there is a countable basis for the **topology** (i.e., collection of open sets) of the space.

Definition

A **coordinate chart** for a topological n-manifold M^n is a pair (U, ϕ) where

$U =$ open set in M, and

$\phi : U \to U' \subset \mathbb{R}^n$ is a homeomorphism from U to the open set U' in $\mathbb{R}^n$.

A collection of coordinate charts whose domains cover the manifold M is said to be an **atlas** for M.

If (U_1, ϕ_1) and (U_2, ϕ_2) are coordinate charts for M with $U_1 \cap U_2$ nonempty, then the map

$$\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$$

is called a **change of coordinates map**, or simply, a **transition map**.

If it is possible to choose an atlas for the topological manifold M^n such that all transition maps are smooth, then M^n is called a **smooth manifold** and the given atlas is called a "**smooth structure**" for M^n.

The Plan: Give and smooth structure on a smooth manifold M, define a function $f : M \rightarrow \mathbb{R}$ to be smooth if and only if $f \circ \phi^{-1}$ is smooth in the sense of ordinary calculus for each coordinate chart (U, ϕ).

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Example: Two atlases on $\mathbb{R}^n$

$$\mathcal{A}_1 = \{(\mathbb{R}^n, Id_{\mathbb{R}^n})\}$$

$$\mathcal{A}_2 = \{(B_1(x), Id_{B_1(x)})\}$$

A smooth atlas $\mathcal{A}$ on M is **maximal** if it is not contained in any strictly larger smooth atlas. In other words, if $\mathcal{A}$ is maximal, then any any chart that is smoothly compatible with every chart in $\mathcal{A}$ is already in $\mathcal{A}$.

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Theorem

Let M be a topological manifold. Every smooth atlas for M is contained in a unique maximal smooth atlas.

Hence, a smooth structure on a topological *n*-manifold is a maximal smooth atlas.

Proof: Let $\mathcal{A}$ be a smooth atlas for M.

Let $\overline{\mathcal{A}}$ denote the set of all charts that are smoothly compatible with every chart in $\mathcal{A}$ (every transition map is smooth).

Step 1: $\overline{\mathcal{A}}$ is smooth.

$\overline{\mathcal{A}}$ is our candidate maximal smooth atlas. We must show any two charts in $\overline{\mathcal{A}}$ are smoothly compatible.

Let $(U, \phi), (V, \psi) \epsilon \overline{\mathcal{A}}$ and let $x = \phi(p) \epsilon \phi(U \cap V)$.

There is some $(W, \theta) \epsilon \mathcal{A}$ such that $p \epsilon W$.

By definition of $\overline{\mathcal{A}}$, both $\theta \circ \phi^{-1}$ and $\psi \circ \theta^{-1}$ are smooth.

Since $p \epsilon U \cap V \cap W$, then $\psi \circ \phi^{-1} = (\psi \circ \theta^{-1}) \circ (\theta \circ \phi^{-1})$ is smooth on a nbh of x.

Hence, $\psi \circ \phi^{-1}$ is smooth on on a nbh of each point in $\phi(U \cap V)$.

Therefore $\overline{\mathcal{A}}$ is a smooth atlas.

Step 2: $\overline{\mathcal{A}}$ is maximal.
Any chart that is smoothly compatible with every chart in $\overline{\mathcal{A}}$ is automatically smoothly compatible with every chart in $\mathcal{A}$, hence, already in $\overline{\mathcal{A}}$.

Step 3: $\overline{\mathcal{A}}$ is unique.
If $\mathcal{B}$ is any other maximal smooth atlas containing $\mathcal{A}$, each of its charts are smoothly compatible with each chart in $\mathcal{A}$ so $\mathcal{B} \subset \overline{\mathcal{A}}$.

By maximality of $\mathcal{B}$, $\mathcal{B} = \overline{\mathcal{A}}$.

Remark

Not every topological manifold can be given a smooth structure. The first example was a 10-dimensional manifold found by Michel Kervaire in 1960.

When we speak of a smooth manifold M^n, we always mean "M^n with a specific choice of smooth structure".

Theorem

Let M be a set, and suppose we are given a collection $\{U_\alpha\}$ of subsets of M, together with and injective map $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ for each α, such that the following properties are satisfied:

1. *For each α, $\phi_\alpha(U_\alpha)$ is open subset of $\mathbb{R}^n$.*

2. *For each α and β, $\phi_\alpha(U_\alpha \cap U_\beta)$ and $\phi_\beta(U_\alpha \cap U_\beta)$ are open subsets of $\mathbb{R}^n$*

3. *Whenever $U_\alpha \cap U_\beta \neq \emptyset$, $\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$ is a diffeomorphism.*

4. *Countably many of the sets U_α cover M.*

5. *Whenever p and q are distinct points in M, either there exists some U_α containing both p and q, or there exists disjoint sets U_α and U_β with $p \epsilon U_\alpha$ and $q \epsilon U_\beta$.*

Then M has a unique smooth manifold structure such that each (U_α, ϕ_α) is a smooth chart.

Proof: Define the topology by taking all sets of the form $\phi_\alpha^{-1}(V)$, where V is an open set of $\mathbb{R}^n$, as a basis.(Check that this is indeed a basis of a topology.)

The ϕ_α are homeomorphisms, so M is locally Euclidean.

By 4, M has a countable basis.

By 5, M is Hausdorff.

Finally, 3 guarantees that $\{(U_\alpha, \phi_\alpha)\}$ is a smooth atlas.

Hence, M is a smooth manifold.

Definition

An n-dimensional topological manifold with boundary is a second countable Hausdorff space M in which every point has a nbh homeomorphic to a relatively open subset of $\mathbb{H}^n$.

Definition

A map $f : A \to \mathbb{R}^k$ where $A \subset \mathbb{R}^n$ is smooth if f admits a smooth extension to an open nbh of each point in A.

Hence, If $U \subset \mathbb{H}^n$ is open, a map $F : U \to \mathbb{R}^k$ is smooth if for each $x \epsilon U$ there exists an open set V in R^n and a smooth map $\widetilde{F} : V \to \mathbb{R}^k$ that agrees with f on $V \cap \mathbb{H}^n$.

Definition

A smooth manifold with boundary is the pair $(M, \mathcal{A})$ where M is a topological manifold and $\mathcal{A}$ is a maximal smooth atlas

Note: The transition maps are smooth in the above sense.

Definition

If M is a smooth n-manifold, a function $f : M \to \mathbb{R}^k$ is said to be **smooth** if for every $p \epsilon M$, there exists a smooth chart (U, ϕ) for M whose domain contains p and such that the composite function $f \circ \phi^{-1}$ is smooth on the open set $\widetilde{U} = \phi(U) \subset \mathbb{R}^n$.

Definition

Let M and N be smooth manifolds, and let $F : M \to N$ be any map. We say that F is a **smooth map** if for every $p \epsilon M$, there exist smooth charts (U, ϕ) containing p and (V, ψ) containing $F(p)$ such that $F(U) \subset V$ and the composite map $\psi \circ F \circ \phi^{-1}$ is sooth from $\phi(U)$ to $\psi(V)$

Let $f : M^n \to N^n$ be a smooth map between smooth manifolds, and suppose there is a smooth map $g : N^n \to M^n$ which is inverse to f, meaning that $g \circ f = identity : M \to M$ and $f \circ g = identity : N \to N$. Then the maps f and g are said to be **diffeomorphisms**, and the manifolds M and N are **diffeomorphic**.

A diffeomorphism is automatically a homeomorphism, but not vice versa.

Diffeomorphic smooth manifolds are automatically homeomorphic.

But John Milnor showed in 1956 that homeomorphic smooth manifolds need not be diffeomorphic, and in fact showed that there are exactly 28 different smooth manifolds which are homeomorphic to the 7-sphere S7 . For this, he won the Fields Medal in 1962.

Partitions of unity on differentiable manifolds.

1. A covering $\{U\}$ of a topological space X is said to be **locally finite** if each point of X has a neighborhood which intersects only finite many sets in the covering.

2. A covering $\{U\}$ of X is said to be a refinement of a covering $\{V\}$ of X if for each set U in the first covering, there is a set V in the second covering such that $U \subset V$.

3. A topological space X is said to be paracompact if every open covering of X has a locally finite refinement.

Definition

The **support** of a function $f : X \to \mathbb{R}$ is the closure of the set of points $x \epsilon X$ where $f(x) \neq 0$.

Definition

Let X be a topological space and $\mathcal{U} = \{U_\alpha\}_{\alpha \epsilon A}$ an opening covering of X. Then a **partition of unity subordinate to** $\mathcal{U}$ is a collection of continuous functions $\psi_\alpha : X \to \mathbb{R}_{\alpha \epsilon A}$ such that

1. $0 \leq \psi_\alpha(x) \leq 1$ for all $\alpha \epsilon A$ and $x \epsilon X$.

2. supp $\psi_\alpha \subset U_\alpha$

3. The set of supports $\{supp \ \psi_\alpha\}_{\alpha \epsilon A}$ is locally finite.

4. $\Sigma_{\alpha \epsilon A} \psi_\alpha(x) = 1$ for all $x \epsilon X$.

If M is a smooth manifold and each function $\psi : M \to \mathbb{R}$ is smooth, then we speak of a smooth partition of unity.

Next Time

Theorem

(Whitney Embedding Theorem) Every smooth m-dimensional manifold admits a smooth embedding into Euclidean space $\mathbb{R}^{2m+1}$.